national accelerator laboratory

TM-313
0402

ORBIT AND FOCUSING BUMPS

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The effect of a general orbit bump $\Delta B(z)$ on the closed orbit and the effect of a general focusing bump $\Delta B'(z)$ on the β-function were given by Courant and Snyder (Ann. of Phys. 3, 1-48, 1958, hereafter referred to as C & S.) Here, we put their formulas into easy-to-apply forms and apply them to the main ring. For completeness, we will outline the derivations of the formulas given in C & S.

I. ORBIT BUMP

With an orbit bump $\Delta B(z)$ the orbit equation is

$$\frac{d^2x}{dz^2} + K(z)x = -\frac{\Delta B}{B\rho}. \tag{1}$$

After the Floquet transformation

$$x = \sqrt{\nu\beta}\, u \qquad dz = \nu\beta\, d\theta \tag{2}$$

we get

$$\frac{d^2u}{d\theta^2} + \nu^2 u = -\frac{\Delta B}{B\rho}(\nu\beta)^{3/2} \equiv F(\theta). \tag{3}$$

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The periodic solution (closed orbit) is

$$u = \frac{1}{4\nu \sin\pi\nu}\left[e^{i\nu(\theta+\pi)}\int_{\theta}^{\theta+2\pi} F(\theta')e^{-i\nu\theta'}d\theta'\right.$$

$$\left. + e^{-i\nu(\theta+\pi)}\int_{\theta}^{\theta+2\pi} F(\theta')e^{i\nu\theta'}d\theta\right]. \qquad (4)$$

[Except for difference in notation this is identical to Eq. (4.7) of C & S]. It is useful to calculate the "invariant" W [Eq. (3.22) of C & S], except now, $W = W(\theta)$ is a function of θ.

$$W(\theta) = \nu\left[u^2 + \frac{1}{\nu^2}\left(\frac{du}{d\theta}\right)^2\right]$$

$$= \frac{\nu}{4\nu^2 \sin^2\pi\nu}\left(\int_{\theta}^{\theta+2\pi} F(\theta')e^{i\nu\theta'}d\theta'\right)\left(\int_{\theta}^{\theta+2\pi} F(\theta')e^{-i\nu\theta'}d\theta'\right)$$

$$= \frac{1}{4\sin^2\pi\nu}\left(\int_{z}^{z+L}\left(\frac{\Delta B}{B\rho}\right)\sqrt{\beta}\ e^{i\phi}dz'\right)\left(\int_{z}^{z+L}\left(\frac{\Delta B}{B\rho}\right)\sqrt{\beta}\ e^{-i\phi}dz'\right) \qquad (5)$$

where

$$\begin{cases} L = \text{orbit length all around} \\ \phi \equiv \int \frac{dz}{\beta} = \text{phase of betatron oscillation.} \end{cases}$$

If the bumps are all localized δ-functions we have

$$\boxed{W(\theta) = \frac{1}{4\sin^2\pi\nu}\left(\sum_n \delta_n\sqrt{\beta_n}\ e^{i\phi_n}\right)\left(\sum_n \delta_n\sqrt{\beta_n}\ e^{-i\phi_n}\right)} \qquad (6)$$

where

$$\delta_n \equiv \frac{(\Delta B\ell)_n}{B\rho} = \text{kick angle of the nth bump.}$$

In between two bumps W is constant and the upper-bound of the orbit displacement in that region is given by $\hat{x} = \sqrt{W\beta}$.

Case 1

If we have only one bump (n = 0)

$$W = \frac{\beta_o \delta_o^{\,2}}{4\sin^2 \pi\nu}.$$

For the main ring if one bending magnet ($\delta_o = 0.0081$) at $\beta_o \cong 90$m is missing we have, since $\sin \pi\nu \cong 1/\sqrt{2}$

$$W \cong \frac{1}{2}\ (90\ \text{m})(0.0081)^2 = 2950\ \text{mm-mrad.}$$

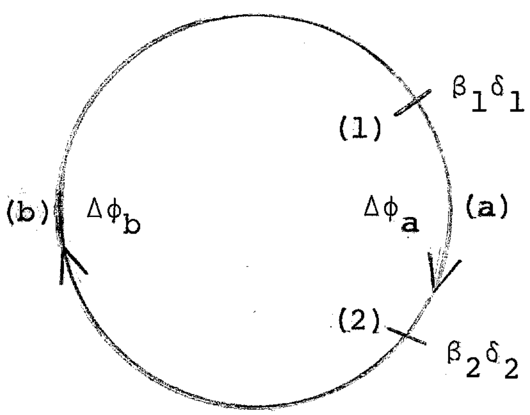
The maximum β is $\beta_{max} \cong 100$ m. This value of W gives for the maximum upper-bound of the closed-orbit displacement

$$\hat{x}_{max} = \sqrt{W\beta_{max}} = 540\ \text{mm}$$

which is, of course, much too large.

Case 2

If we have two bumps W(θ) has only two values W_a and W_b in region (a) (from bump 1 to bump 2) and region (b) (from bump 2 to bump 1), respectively. They are



$$\left.\begin{aligned} W_a &= \frac{1}{4\sin^2\pi\nu}\left(\delta_2\sqrt{\beta_2}\, e^{i\phi_2} + \delta_1\sqrt{\beta_1}\, e^{i\phi_1}\right) \times \\ &\qquad\qquad \left(\delta_2\sqrt{\beta_2}\, e^{-i\phi_2} + \delta_1\sqrt{\beta_1}\, e^{-i\phi_1}\right) \\ &= \frac{1}{4\sin^2\pi\nu}\left(\delta_1^2\beta_1 + 2\delta_1\delta_2\sqrt{\beta_1\beta_2}\cos\Delta\phi_b + \delta_2^2\beta_2\right) \\ W_b &= \frac{1}{4\sin^2\pi\nu}\left(\delta_1^2\beta_1 + 2\delta_1\delta_2\sqrt{\beta_1\beta_2}\cos\Delta\phi_a + \delta_2^2\beta_2\right). \end{aligned}\right\} \quad (8)$$

This corresponds to the formulas given in TM-294-Eq.(2). For example, if $\Delta\phi_a = \pi$ and $\delta_1\sqrt{\beta_1} = \delta_2\sqrt{\beta_2}$ we have $W_b = 0$. This is the case of a local orbit bump formed by two magnets π-phase advance apart.

II. FOCUSING BUMP

With a focusing bump $\Delta B'(z)$ the β-deviation equation after the Floquet transformation is

$$\frac{d^2}{d\theta^2}\left(\frac{\Delta\beta}{\beta}\right) + 4\nu^2\left(\frac{\Delta\beta}{\beta}\right) = -2\frac{\Delta B'}{B\rho}(\nu\beta)^2 \equiv G(\theta). \qquad (9)$$

The periodic solution is

$$\frac{\Delta\beta}{\beta} = \frac{1}{8\nu\sin 2\pi\nu}\left[e^{2i\nu(\theta+\pi)}\int_\theta^{\theta+2\pi} G(\theta')e^{-2i\nu\theta'}d\theta' + e^{-2i\nu(\theta+\pi)}\int_\theta^{\theta+2\pi} G(\theta')e^{2i\nu\theta'}d\theta'\right]. \qquad (10)$$

(Except for difference in notation this is identical to Eq. (4.50) of C & S). We can define a similar "invariant" $U = U(\theta)$ by

$$U(\theta) = \left(\frac{\Delta\beta}{\beta}\right)^2 + \frac{1}{4\nu^2}\left(\frac{d}{d\theta}\frac{\Delta\beta}{\beta}\right)^2$$

$$= \frac{1}{16\nu^2\sin^2 2\pi\nu}\left(\int_\theta^{\theta+2\pi} G(\theta')e^{2i\nu\theta'}d\theta'\right) x$$

$$\left(\int_\theta^{\theta+2\pi} G(\theta')e^{-2i\nu\theta'}d\theta'\right)$$

$$= \frac{1}{4\sin^2 2\pi\nu}\left(\int_z^{z+L}\left(\frac{\Delta B'}{B\rho}\right)\beta e^{-2i\phi}dz'\right)\left(\int_z^{z+L}\left(\frac{\Delta B'}{B\rho}\right)\beta e^{-2i\phi}dz'\right). \quad (11)$$

If the bumps are localized δ-functions we have

$$\boxed{U(\theta) = \frac{1}{4\sin^2 2\pi\nu}\left(\sum_n \varepsilon_n\beta_n e^{2i\phi_n}\right)\left(\sum_n \varepsilon_n\beta_n e^{-2i\phi_n}\right)} \quad (12)$$

where

$$\varepsilon_n \equiv \frac{(\Delta B'\ell)_n}{B\rho} = \text{focusing "kink" of the nth bump.}$$

In between two bumps U is a constant and the upper-bound of $\Delta\beta$ in that region is $\widehat{\Delta\beta} = \beta\sqrt{U}$.

Case 1

If we have only one bump $(n = 0)$

$$U = \frac{\beta_o^2\varepsilon_o^2}{4\sin^2 2\pi\nu} . \quad (13)$$

For the main ring if one focusing quadrupole ($\varepsilon_o = 0.040\ m^{-1}$) at, say, $\beta_{ox} \cong 99m$ and $\beta_{oy} \cong 27\ m$ is missing, we have, since $\sin 2\pi\nu \cong 1$

$$\begin{cases} U_x \cong \frac{1}{4}\ (99m)^2 (0.040\ m^{-1})^2 = 3.92 \\ U_y \cong \frac{1}{4}\ (27m)^2 (0.040\ m^{-1})^2 = 0.29. \end{cases}$$

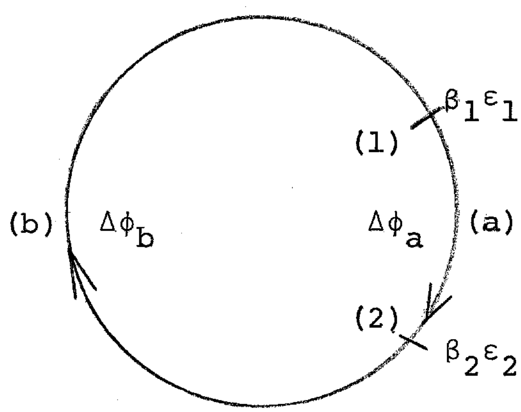
The upper-bounds of the increased β-functions, namely, $B + \widehat{\Delta\beta}$ are, then

$$\begin{cases} (\beta + \widehat{\Delta\beta})_x = (1 + \sqrt{U_x})\beta_x \cong 3.0\ \beta_x \\ (\beta + \widehat{\Delta\beta})_y = (1 + \sqrt{U_y})\beta_y \cong 1.5\ \beta_y. \end{cases} \tag{14}$$

Since the main ring aperture is rather large these increases in β may well be tolerable.

Case 2

With two bumps $U(\theta)$ has two values U_a and U_b in regions (a) and (b), respectively. In this case we have



$$\begin{cases} U_a = \dfrac{1}{4\sin^2 2\pi\nu}\left(\varepsilon_2\beta_2 e^{2i\phi_2} + \varepsilon_1\beta_1 e^{2i\phi_1}\right) \times \\ \qquad\qquad\qquad\qquad \left(\varepsilon_2\beta_2 e^{-2i\phi_2} + \varepsilon_1\beta_1 e^{-2i\phi_1}\right) \\ \quad = \dfrac{1}{4\sin^2 2\pi\nu}\left(\varepsilon_1^2\beta_1^2 + 2\varepsilon_1\varepsilon_2\beta_1\beta_2 \cos\Delta\phi_b + \varepsilon_2^2\beta_2^2\right) \\ U_b = \dfrac{1}{4\sin^2 2\pi\nu}\left(\varepsilon_1^2\beta_1^2 + 2\varepsilon_1\varepsilon_2\beta_1\beta_2 \cos\Delta\phi_a + \varepsilon_2^2\beta_2^2\right). \end{cases} \tag{15}$$

Suppose we ask the question whether it is possible to at least partially compensate for a missing quadrupole by turning off a second quadrupole.

U is zero only when $\varepsilon_1\beta_1 = \varepsilon_2\beta_2$ and $\cos\Delta\phi = -1$. For a quadrupole $|\varepsilon|$ has the same value in the x and the y planes. To compensate equally for both planes we should have $\beta_2 = \beta_1$, namely if a focusing quadrupole is missing we should turn off also a focusing quadrupole. Furthermore, since $\Delta\phi_a + \Delta\phi_b = 2\pi\nu \cong 2\pi(20\frac{1}{4})$ to get $\cos\Delta\phi_a$ and $\cos\Delta\phi_b$ equally negative so that U_a and U_b are equally small we should have $\Delta\phi_a = 2\pi(k + \frac{5}{8})$ and $\Delta\phi_b = 2\pi(19 - k + \frac{5}{8})$ with k = integer. Then $\cos\Delta\phi_a = \cos\Delta\phi_b = -\cos\frac{\pi}{4} = -\frac{1}{\sqrt{2}}$. And we have

$$U_a = U_b \cong \frac{2-\sqrt{2}}{4}\,\varepsilon_1^2\beta_1^2 = 0.146\,\varepsilon_1^2\beta_1^2$$

$$= \begin{cases} 2.30 & \text{x-plane} \\ 0.17 & \text{y-plane} \end{cases}$$

if the first missing quadrupole is a focusing quadrupole. Thus the increase in β is reduced to

$$\begin{cases} (\beta + \widehat{\Delta\beta})_x \cong 2.5\ \beta_x \\ (\beta + \widehat{\Delta\beta})_y \cong 1.4\ \beta_y. \end{cases} \tag{16}$$

Comparing these values with those in Eq. (14) we see that the effect of a missing quadrupole can indeed be partially compensated by turning off another quadrupole, but the amount of compensation is not very large. Here we considered only a compromised compensation in both the x and the y planes and in both regions (a) and (b). It is possible to improve the compensation if for some reason only one of the planes or one of the regions is considered important.

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ERRATUM AND ADDENDUM TO TM-313

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Both Eqs. (1) and (9) are approximate equations. For Eq. (1) the approximation assumes that $\frac{x}{\beta} << 1$ and terms of the second and higher orders in $\frac{x}{\beta}$ are neglected. These conditions are satisfied for the example cases given on pp. 3 and 4.

For Eq. (9) the approximation assumes that $\frac{\Delta\beta}{\beta} << 1$ and $\frac{\Delta B'}{B'} << 1$, and terms of the second and higher orders in $\frac{\Delta\beta}{\beta}$ and $\frac{\Delta B'}{B'}$ are neglected. These conditions are <u>not</u> satisfied for the example cases given on pp. 6 and 7. The results are, therefore, <u>invalid</u>.

Eq. (10) shows that $\frac{\Delta\beta}{\beta}$ (if << 1) is a sinusoidal function of θ with amplitude $\sqrt{U}$. For $\sqrt{U} > 1$, then, at some θ-locations $\frac{\Delta\beta}{\beta} < -1$ and the modified $\overline{\beta} = \beta + \Delta\beta < 0$ which is certainly not meaningful. This is another indication that Eq. (9) and its solution Eq. (10) are invalid when $\frac{\Delta\beta}{\beta} = \sqrt{U} > 1$.

For the case of one δ-function focusing bump the exact solution can be obtained using the transfer matrix. The transfer matrix around the entire closed orbit plus the bump (ε_o) is

$$\begin{pmatrix} 1 & 0 \\ -\varepsilon_o & 1 \end{pmatrix} \left[\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \cos 2\pi\nu + \begin{pmatrix} \alpha_o & \beta_o \\ -\gamma_o & -\alpha_o \end{pmatrix} \sin 2\pi\nu \right]$$

$$= \begin{pmatrix} 1 & 0 \\ -\varepsilon_o & 1 \end{pmatrix} \cos 2\pi\nu + \begin{pmatrix} \alpha_o & \beta_o \\ -(\gamma_o+\varepsilon_o\alpha_o) & -(\alpha_o+\varepsilon_o\beta_o) \end{pmatrix} \sin 2\pi\nu$$

where, as before, $\varepsilon_o \equiv \frac{(\Delta B'\ell)_o}{B\rho}$. The modified "tune" $\bar{\nu}$ and β-function at the bump $\bar{\beta}_o$ are, therefore, given by

$$\begin{cases} \cos 2\pi\bar{\nu} = \cos 2\pi\nu - \frac{\varepsilon_o\beta_o}{2} \sin 2\pi\nu \\ \bar{\beta}_o \sin 2\pi\bar{\nu} = \beta_o \sin 2\pi\nu . \end{cases} \tag{1A}$$

As ε_o varies from zero to either positive or negative values stability limits $\cos 2\pi\bar{\nu} = \pm 1$ will be encountered at certain values of ε_o. Beyond these values of ε_o, $|\cos 2\pi\bar{\nu}| > 1$ and the motion is unstable. At the stability limits the modified β-function $\bar{\beta}$ is ∞ everywhere except at discrete θ-locations where $\bar{\beta} = 0$, namely $\frac{\Delta\beta}{\beta} \equiv \frac{\bar{\beta}-\beta}{\beta}$ is ∞ everywhere except at these discrete θ-locations where $\frac{\Delta\beta}{\beta} = -1$. Although at the stability limit this exact $\frac{\Delta\beta}{\beta}$ is hardly sinusoidal, one may expect that the stability limits correspond roughly to $\sqrt{U} = 1$ when the "approximate" $\bar{\beta}$ as given by Eq. (10) also goes to zero at these discrete θ-locations. Eq. (13) gives, then, for the stability limits

$$\frac{\varepsilon_o\beta_o}{2} = \pm \sin 2\pi\nu \quad \text{"approximate"} \tag{2A}$$

while the exact conditions are given by Eq. (1A) as

$$\frac{\varepsilon_o\beta_o}{2} = \frac{\cos 2\pi\nu \mp 1}{\sin 2\pi\nu}$$

$$= - \frac{1}{\cos 2\pi\nu \pm 1} \sin 2\pi\nu. \quad \text{exact} \tag{3A}$$

The exact and the "approximate" conditions are identical when ν = (integer) $\pm \frac{1}{4}$.

For the main ring $\nu \cong 20\frac{1}{4}$. Both Eqs. (2A) and (3A) give for the stability limits

$$\frac{\varepsilon_o\beta_o}{2} = \pm 1$$

or, for $\beta_o \cong 100$ m

$$\varepsilon_o = \pm\frac{2}{\beta_o} \cong \pm 0.02 \text{ m}^{-1}.$$

Missing one quadrupole ($\varepsilon_o = \pm 0.04$ m^{-1}) will take us <u>beyond the stability limit</u>. The most we can tolerate is missing $\frac{1}{2}$ of a quadrupole.

The "invariant" U is clearly also an approximate invariant valid only when $\frac{\Delta\beta}{\beta} << 1$. We can put U in a more conventional form.

$$U = \left(\frac{\Delta\beta}{\beta}\right)^2 + \frac{1}{4\nu^2}\left[\frac{d}{d\theta}\left(\frac{\Delta\beta}{\beta}\right)\right]^2$$

$$= \left(\frac{\Delta\beta}{\beta}\right)^2 + \frac{1}{4}\left[\beta\frac{d}{dz}\left(\frac{\Delta\beta}{\beta}\right)\right]^2$$

$$= \left(\frac{\Delta\beta}{\beta}\right)^2 + \left[-\frac{\beta'}{2}\frac{\Delta\beta}{\beta} + \frac{(\Delta\beta)'}{2}\right]^2$$

$$= \left(\frac{\Delta\beta}{\beta}\right)^2 + \alpha^2 \left(\frac{\Delta\beta}{\beta} - \frac{\Delta\alpha}{\alpha}\right)^2 \tag{4A}$$

where prime means $\frac{d}{dz}$ and $\alpha = -\frac{\beta'}{2}$, $\Delta\alpha = -\frac{(\Delta\beta)'}{2}$.

D. A. Edwards gave the exact form of this invariant as

$$U = \frac{\left(\frac{\Delta\beta}{\beta}\right)^2 + \alpha^2 \left(\frac{\Delta\beta}{\beta} - \frac{\Delta\alpha}{\alpha}\right)^2}{1 + \frac{\Delta\beta}{\beta}}. \tag{5A}$$

His derivation is given below: Consider two locations 1 and 2 around the closed orbit with no focusing bump in between. The transfer matrices from locations 1 and 2 all the way around the closed orbit are respectively

$$\begin{aligned} \overline{M}_1 &= \cos 2\pi\overline{\nu} + \overline{J}_1 \sin 2\pi\overline{\nu} \\ &= \cos 2\pi\overline{\nu} + (J_1+\Delta J_1) \sin 2\pi\overline{\nu} \end{aligned}$$

and

$$\begin{aligned} \overline{M}_2 &= \cos 2\pi\overline{\nu} + \overline{J}_2 \sin 2\pi\overline{\nu} \\ &= \cos 2\pi\overline{\nu} + (J_2+\Delta J_2) \sin 2\pi\overline{\nu}. \end{aligned}$$

Writing the transfer matrix from location 1 to location 2 as M_{12} (there is no need for a bar on top because there is no bump between locations 1 and 2) the relation $\overline{M}_2 = M_{12}M_1M_{12}^{-1}$ leads to

$$J_2+\Delta J_2 = M_{12}(J_1+\Delta J_1)M_{12}^{-1}.$$

Remembering that $J_2 = M_{12}J_1M_{12}^{-1}$ we get

$$\Delta J_2 = M_{12}\Delta J_1 M_{12}^{-1}$$

which shows that the determinant of ΔJ is invariant within a bump-free region. We can, thus, write

$$U = -|\Delta J| = (\Delta\alpha)^2 - (\Delta\beta)(\Delta\gamma) = \text{invariant.}$$

Substituting

$$\Delta\gamma = \frac{1+(\alpha+\Delta\alpha)^2}{\beta+\Delta\beta} - \frac{1+\alpha^2}{\beta}$$

$$= - \frac{\frac{\Delta\beta}{\beta} + \alpha^2 \left[\frac{\Delta\beta}{\beta} - 2\frac{\Delta\alpha}{\alpha} - \left(\frac{\Delta\alpha}{\alpha}\right)^2\right]}{\beta\left(1+\frac{\Delta\beta}{\beta}\right)}$$

we get directly the expression (5A).

I am grateful to Dr. S. Ohnuma for pointing out the error in TM-313 and to Dr. D. Edwards for the derivation of the exact expression of the invariant U, and to both of them for several illuminating discussions.